On Certain Metrical Properties of Surfaces.

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First consider a surface in a space of $n+1$ dimensions. For brevity in speaking of spaces of any number of dimensions I shall use the symbol (commonly employed) M_{n+1}; this is taken to denote a space of $n+1$ dimensions. A surface in M_{n+1} is understood to be expressed by a single relation between the $n+1$ coördinates which determine the position of a point in M_{n+1}. Denoting these coördinates by x_i where $i = 1, 2, 3 \ldots n+1$, the equation

$$\Phi(x_1 x_2 \ldots x_{n+1}) = 0$$

is the equation of an n-dimensional surface in M_{n+1}. Since we have one relation connecting the $n+1$ quantities x_i, each of these may be given as a function of n independent variables $u_1 u_2 \ldots u_n$.

Denote by $a_k^{(i)}$ the differential coefficient of x_i with respect to u_k, then

$$dx_i = a_1^{(i)} du_1 + a_2^{(i)} du_2 + \ldots + a_n^{(i)} du_n ;$$

also denote by $a_{jk}^{(i)}$ the second differential coefficient of x_i with respect to u_j and u_k, then

$$d^2 x_i = a_{11}^{(i)} du_1^2 + 2a_{12}^{(i)} du_1 du_2 + 2a_{13}^{(i)} du_1 du_3 + \ldots + a_{nn}^{(i)} du_n^2 .$$

For the element of length ds of a curve traced on $\Phi = 0$, we have then

$$ds^2 = \sum_{i=1}^{i=n+1} a_1^{(i)^2} du_1^2 + \Sigma a_2^{(i)^2} du_2^2 + \ldots + \Sigma a_n^{(i)^2} du_n^2$$
$$+ 2\Sigma a_1^{(i)} a_2^{(i)} du_1 du_2 + \ldots + 2\Sigma a_{n-1}^{(i)} a_n^{(i)} du_{n-1} du_n .$$

The limits of the summation have only been written once, as they are of course the same for every term. For brevity, write

$$E_{ii} = a_i^{(1)^2} + a_i^{(2)^2} + \ldots + a_i^{(n+1)^2}$$
$$E_{jk} = a_j^{(1)} a_k^{(1)} + a_j^{(2)} a_k^{(2)} + \ldots + a_j^{(n)} a_k^{(n)} ;$$

then for ds we have

$$ds^2 = E_{11}du_1^2 + E_{12}du_1du_2 + \ldots + E_{jk}du_jdu_k + \ldots + E_{nn}du_n^2 .$$

From the quantities $a_j^{(i)}$ form the determinants of which

$$\begin{vmatrix} a_1^{(2)} & a_1^{(3)} & \ldots & a_1^{(n+1)} \\ a_2^{(2)} & a_2^{(3)} & \ldots & a_2^{(n+1)} \\ \cdots & \cdots & \cdots & \cdots \\ a_n^{(2)} & a_n^{(3)} & \ldots & a_n^{(n+1)} \end{vmatrix} = A_1$$

is the first. There will be $n+1$ of these determinants each of the degree n; squaring these and adding, we obtain, as is well known, a symmetrical determinant of the degree n, viz.

$$V^2 = \sum_1^n A_i^2 = \begin{vmatrix} E_{11} & E_{12} & E_{13} & \ldots & E_{1n} \\ E_{21} & E_{22} & E_{23} & \ldots & E_{2n} \\ E_{31} & E_{32} & E_{33} & \ldots & E_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ E_{n1} & E_{n2} & E_{n3} & \ldots & E_{nn} \end{vmatrix};$$

of course $E_{ij} = E_{ji}$.

The differential equation of the surface $\Phi = 0$ can now be written as

$$A_1dx_1 + A_2dx_2 + \ldots + A_{n+1}dx_{n+1} = 0 .$$

The direction-cosines of the normal to this surface at any point are

$$\alpha_1, \alpha_2, \ldots \alpha_{n+1} = \frac{A_1}{V}, \frac{A_2}{V}, \ldots \frac{A_{n+1}}{V}.$$

The element of area of the surface (as will be shown hereafter) is

$$dS = Vdu_1du_2 \ldots du_n ,$$

or by virtue of the above relations,

$$dS = (\alpha_1A_1 + \alpha_2A_2 + \ldots + \alpha_{n+1}A_{n+1}) \prod_1^n du_i .$$

Suppose we have now a second surface

$$\Psi(y_1y_2 \ldots y_{n+1}) = 0$$

connected with the first by the relations

$$y_i = f_i(x_1x_2 \ldots x_{n+1}) \qquad i = 1, 2 \ldots n+1 .$$

Denote by $\beta_1, \beta_2, \ldots \beta_{n+1}$ the direction-cosines of the normal to this second surface, and by $B_1, B_2 \ldots B_{n+1}$ what the determinants A_i of the first surface become for the second surface; then writing

$$U^2 = B_1^2 + B_2^2 + \ldots + B_{n+1}^2$$

we have
$$\beta_i = \frac{B_i}{U}$$

and for the element of area
$$d\Sigma = (\beta_1 B_1 + \beta_2 B_2 + \ldots + \beta_{n+1} B_{n+1}) \prod_1^n du_i .$$

The ratio between the elements of area dS and $d\Sigma$ is
$$\frac{d\Sigma}{dS} = \frac{U}{V}.$$

Now V may obviously be written in the form

$$V = \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_{n+1} \\ \frac{dx_1}{du_1} & \frac{dx_2}{du_1} & \frac{dx_3}{du_1} & \ldots & \frac{dx_{n+1}}{du_1} \\ \frac{dx_1}{du_2} & \frac{dx_2}{du_2} & \frac{dx_3}{du_2} & \ldots & \frac{dx_{n+1}}{du_2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{dx_1}{du_n} & \frac{dx_2}{du_n} & \frac{dx_3}{du_n} & \ldots & \frac{dx_{n+1}}{du_n} \end{vmatrix},$$

and similarly, denoting by $\left(\frac{dy_i}{du_k}\right)$ the quantity

$$\frac{dy_i}{dx_1}\frac{dx_1}{du_k} + \frac{dy_i}{dx_2}\frac{dx_2}{du_k} + \ldots + \frac{dy_i}{dx_{n+1}}\frac{dx_{n+1}}{du_k},$$

we may write

$$U = \begin{vmatrix} \beta_1 & \beta_2 & \beta_3 & \ldots & \beta_{n+1} \\ \left(\frac{dy_1}{du_1}\right) & \left(\frac{dy_2}{du_1}\right) & \left(\frac{dy_3}{du_1}\right) & \ldots & \left(\frac{dy_{n+1}}{du_1}\right) \\ \left(\frac{dy_1}{du_2}\right) & \left(\frac{dy_2}{du_2}\right) & \left(\frac{dy_3}{du_2}\right) & \ldots & \left(\frac{dy_{n+1}}{du_2}\right) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \left(\frac{dy_1}{du_n}\right) & \left(\frac{dy_2}{du_n}\right) & \left(\frac{dy_3}{du_n}\right) & \ldots & \left(\frac{dy_{n+1}}{du_n}\right) \end{vmatrix}$$

The direction-cosines of the lines of intersection of the surfaces u taken $n-1$ at a time with the surface Φ are

$$\frac{1}{\sqrt{E_{11}}}\frac{dx_1}{du_1},\ \frac{1}{\sqrt{E_{11}}}\frac{dx_2}{du_1},\ \ldots\ \frac{1}{\sqrt{E_{11}}}\frac{dx_{n+1}}{du_1},$$

and so for the remaining cases. At the common point of intersection of these lines, the normal (α) is at right angles to them all, and so

$$\alpha_1 \frac{dx_1}{du_i} + \alpha_2 \frac{dx_2}{du_i} + \ldots + \alpha_{n+1} \frac{dx_{n+1}}{du_i} = 0, \quad i = 1, 2 \ldots n.$$

Also $\alpha_1^2 + \alpha_2^2 + \ldots + \alpha_{n+1}^2 = 1$. The determinant U may then be replaced by

$$U = \begin{vmatrix} \beta_1 & \beta_2 & \ldots & \beta_{n+1}, & 0 \\ \left(\frac{dy_1}{du_1}\right) & \left(\frac{dy_2}{du_1}\right) & \ldots & \left(\frac{dy_{n+1}}{du_1}\right), & \sum_1^{n+1} \alpha_i \frac{dx_i}{du_1} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \left(\frac{dy_1}{du_n}\right) & \left(\frac{dy_2}{du_n}\right) & \ldots & \left(\frac{dy_{n+1}}{du_n}\right), & \Sigma \alpha_i \frac{dx_i}{du_n} \\ \sum_1^{n+1} \alpha_i \frac{dy_1}{dx_i} & \Sigma \alpha_i \frac{dy_2}{dx_i} & \ldots & \Sigma \alpha_i \frac{dy_{n+1}}{dx_i}, & \Sigma \alpha_i^2 \end{vmatrix}$$

This is however the product of two determinants, viz.

$$U = \begin{vmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & \frac{dx_1}{du_1} & \frac{dx_2}{du_1} & \ldots & \frac{dx_{n+1}}{du_1} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \frac{dx_1}{du_n} & \frac{dx_2}{du_n} & \ldots & \frac{dx_{n+1}}{du_n} \\ 0 & \alpha_1 & \alpha_2 & \ldots & \alpha_{n+1} \end{vmatrix} \begin{vmatrix} \beta_1 & \frac{dy_1}{dx_1} & \frac{dy_1}{dx_2} & \ldots & \frac{dy_1}{dx_{n+1}} \\ \beta_2 & \frac{dy_2}{dx_1} & \frac{dy_2}{dx_2} & \ldots & \frac{dy_2}{dx_{n+1}} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \beta_{n+1} & \frac{dy_{n+1}}{dx_1} & \frac{dy_{n+1}}{dx_2} & \ldots & \frac{dy_{n+1}}{dx_{n+1}} \\ 0 & \alpha_1 & \alpha_2 & \ldots & \alpha_{n+1} \end{vmatrix}$$

The first of these is however equal to V, so we find for the ratio of the corresponding elements of area on the two surfaces,

$$\frac{d\Sigma}{dS} = \frac{U}{V} = \begin{vmatrix} \beta_1 & \frac{dy_1}{dx_1} & \frac{dy_1}{dx_2} & \ldots & \frac{dy_1}{dx_{n+1}} \\ \beta_2 & \frac{dy_2}{dx_1} & \frac{dy_2}{dx_2} & \ldots & \frac{dy_2}{dx_{n+1}} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \beta_{n+1} & \frac{dy_{n+1}}{dx_1} & \frac{dy_{n+1}}{dx_2} & \ldots & \frac{dy_{n+1}}{dx_{n+1}} \\ 0 & \alpha_1 & \alpha_2 & \ldots & \alpha_{n+1} \end{vmatrix},$$

or say for brevity,

$$\frac{d\Sigma}{dS} = \frac{U}{V} = K.$$

This is rather an interesting formula, and one which I have not been able to find given anywhere else; I actually obtained it first for three dimensions, but the generalization was obvious and attended with no difficulty, so for the purposes of another part of the paper I prefer to give it in this form at once. If we denote by x_1, x_2, x_3 the coördinates of a point on an ordinary surface, and by y_1, y_2, y_3 the coördinates of the corresponding point on a second surface, we have*

$$\frac{d\Sigma}{dS} = \frac{U}{V} = \begin{vmatrix} \beta_1 & \frac{dy_1}{dx_1} & \frac{dy_1}{dx_2} & \frac{dy_1}{dx_3} \\ \beta_2 & \frac{dy_2}{dx_1} & \frac{dy_2}{dx_2} & \frac{dy_2}{dx_3} \\ \beta_3 & \frac{dy_3}{dx_1} & \frac{dy_3}{dx_2} & \frac{dy_3}{dx_3} \\ 0 & \alpha_1 & \alpha_2 & \alpha_3 \end{vmatrix}, \text{ say } k.$$

Suppose the second surface to be a sphere of radius unity, whose radii are parallel to the normals at the corresponding point of the first surface. Let the first surface be given in the form

$$x_3 = \phi\,(x_1 x_2)$$

and denote by p and q the first differential coefficients of x_3 with respect to x_1 and x_2; also as usual write r, s, t for the second differential coefficients of x_3 with respect to x_1 and x_2.

The second surface is

$$y_1^2 + y_2^2 + y_3^2 = 1\,;$$

this is satisfied by writing

$$Y_1 = \frac{p}{Q}, \quad Y_2 = \frac{q}{Q}, \quad Y_3 = \frac{-1}{Q}$$

where $Q = \sqrt{1+p^2+q^2}$. The ratio between the corresponding elements of area on the two surfaces is now

$$\begin{vmatrix} \frac{p}{Q}, & \frac{(1+q^2)\,r - pqs}{Q^3}, & \frac{(1+q^2)\,s - pqt}{Q^3}, & 0 \\ \frac{q}{Q}, & \frac{(1+p^2)\,s - pqr}{Q^3}, & \frac{(1+p^2)\,t - pqs}{Q^3}, & 0 \\ \frac{-1}{Q}, & \frac{pr+qs}{Q^3}, & \frac{ps+qt}{Q^3}, & 0 \\ 0, & \frac{p}{Q}, & \frac{q}{Q}, & \frac{-1}{Q} \end{vmatrix}$$

* Some time after the above had gone to press I was informed that Neumann had stated this theorem, without proof, for space of three dimensions, in Volume IX of the *Mathematische Annalen*.—T. C.

This reduces to

$$\frac{-1}{Q^4}\begin{vmatrix} p, & (1+q^2)\,r-pqs, & (1+q^2)\,s-pqt \\ q, & (1+p^2)\,s-pqr, & (1+p^2)\,t-pqs \\ -1, & pr+qs, & ps+qt \end{vmatrix}.$$

Multiply the first row of the determinant by p, the second by q, and the third by -1, and add the first two rows to the last: we have then

$$\frac{1}{Q^2}\begin{vmatrix} (1+q^2)\,r-pqs, & (1+q^2)\,s-pqt \\ (1+p^2)\,s-pqr, & (1+p^2)\,t-pqs \end{vmatrix}.$$

Expanding this we have finally

$$k=\frac{rt-s^2}{(1+p^2+q^2)^2},$$

or calling R and R' the radii of curvature of the first surface,

$$k=\frac{1}{RR'},$$

the measure of curvature, as it should be. The reduction of the determinant in the case where the equation of the first surface is in the form

$$\Phi(x_1x_2x_3)=0$$

is a little more complicated, but attended with no particular difficulty. We of course must arrive at Gauss's expression for the measure of curvature in this case. The quantities $\alpha_1,\ \alpha_2,\ \alpha_3$ are the direction-cosines of the normal to the first surface at the point (x), and $\beta_1,\ \beta_2,\ \beta_3$ are the direction-cosines of the normal to the second surface, or sphere, at the corresponding point (y); then

$$\alpha_1,\ \alpha_2,\ \alpha_3=\beta_1,\ \beta_2,\ \beta_3=\frac{A_1}{V},\ \frac{A_2}{V},\ \frac{A_3}{V};$$

and of course

$$y_1,\ y_2,\ y_3=\frac{A_1}{V},\ \frac{A_2}{V},\ \frac{A_3}{V},$$

since the sphere is of radius unity. Consequently

$$\frac{d\Sigma}{dS}=k=\begin{vmatrix} \frac{A_1}{V}, & \frac{d}{dx_1}\cdot\frac{A_1}{V}, & \frac{d}{dx_2}\cdot\frac{A_1}{V}, & \frac{d}{dx_3}\cdot\frac{A_1}{V} \\ \frac{A_2}{V}, & \frac{d}{dx_1}\cdot\frac{A_2}{V}, & \frac{d}{dx_2}\cdot\frac{A_2}{V}, & \frac{d}{dx_3}\cdot\frac{A_2}{V} \\ \frac{A_3}{V}, & \frac{d}{dx_1}\cdot\frac{A_3}{V}, & \frac{d}{dx_2}\cdot\frac{A_3}{V}, & \frac{d}{dx_3}\cdot\frac{A_3}{V} \\ 0, & \frac{A_1}{V}, & \frac{A_2}{V}, & \frac{A_3}{V} \end{vmatrix}$$

The reduction of this to the ordinary form is quite easy. Write

$$\frac{A_1}{V},\ \frac{A_2}{V},\ \frac{A_3}{V} = \cos\theta_1,\ \cos\theta_2,\ \cos\theta_3,$$

and we have

$$k = \begin{vmatrix} \cos\theta_1, & -\sin\theta_1 \dfrac{d\theta_1}{dx_1}, & -\sin\theta_1 \dfrac{d\theta_1}{dx_2}, & -\sin\theta_1 \dfrac{d\theta_1}{dx_3} \\ \cos\theta_2, & -\sin\theta_2 \dfrac{d\theta_2}{dx_1}, & -\sin\theta_2 \dfrac{d\theta_2}{dx_2}, & -\sin\theta_2 \dfrac{d\theta_2}{dx_3} \\ \cos\theta_3, & -\sin\theta_3 \dfrac{d\theta_3}{dx_1}, & -\sin\theta_3 \dfrac{d\theta_3}{dx_2}, & -\sin\theta_3 \dfrac{d\theta_3}{dx_3} \\ 0, & \cos\theta_1, & \cos\theta_2, & \cos\theta_3, \end{vmatrix}$$

Multiply the first row by $\cos\theta_1$, the second by $\cos\theta_2$, and the third by $\cos\theta_3$, and add the first and second rows to the third. The third row becomes now $1, 0, 0, 0$; interchanging the third and fourth rows, paying attention to all the signs, we have

$$k = \frac{\sin\theta_1 \sin\theta_2}{\cos\theta_3} \begin{vmatrix} \dfrac{d\theta_1}{dx_1} & \dfrac{d\theta_1}{dx_2} & \dfrac{d\theta_1}{dx_3} \\ \dfrac{d\theta_2}{dx_1} & \dfrac{d\theta_2}{dx_2} & \dfrac{d\theta_2}{dx_3} \\ \cos\theta_1 & \cos\theta_2 & \cos\theta_3 \end{vmatrix}$$

Now

$$\frac{d}{dx} = \frac{du_1}{dx}\frac{d}{du_1} + \frac{du_2}{dx}\frac{d}{du_2};$$

changing in the determinant differential coefficients with respect to x into differential coefficients with respect to u_1 and u_2, we have after some easy reductions,

$$k = \frac{1}{\cos\theta_3}\left\{\frac{d\cos\theta_1}{du_1}\frac{d\cos\theta_2}{du_2} - \frac{d\cos\theta_1}{du_2}\frac{d\cos\theta_2}{du_1}\right\}\left\{\cos\theta_1\left(\frac{du_1}{dx_2}\frac{du_2}{dx_3} - \frac{du_1}{dx_3}\frac{du_2}{dx_2}\right)\right.$$
$$\left. + \cos\theta_2\left(\frac{du_1}{dx_3}\frac{du_2}{dx_1} - \frac{du_1}{dx_1}\frac{du_2}{dx_3}\right) + \cos\theta_3\left(\frac{du_1}{dx_1}\frac{du_2}{dx_2} - \frac{du_1}{dx_2}\frac{du_2}{dx_1}\right)\right\}.$$

Denote by E'_{11}, E'_{12}, E'_{22} the determinants

$$\begin{vmatrix} \dfrac{d^2x_1}{du_1^2} & \dfrac{d^2x_2}{du_1^2} & \dfrac{d^2x_3}{du_1^2} \\ \dfrac{dx_1}{du_1} & \dfrac{dx_2}{du_1} & \dfrac{dx_3}{du_1} \\ \dfrac{dx_1}{du_2} & \dfrac{dx_2}{du_2} & \dfrac{dx_3}{du_2} \end{vmatrix},\quad \begin{vmatrix} \dfrac{d^2x_1}{du_1du_2} & \dfrac{d^2x_2}{du_1du_2} & \dfrac{d^2x_3}{du_1du_2} \\ \dfrac{dx_1}{du_1} & \dfrac{dx_2}{du_1} & \dfrac{dx_3}{du_1} \\ \dfrac{dx_1}{du_2} & \dfrac{dx_2}{du_2} & \dfrac{dx_3}{du_2} \end{vmatrix},\quad \begin{vmatrix} \dfrac{d^2x_1}{du_2^2} & \dfrac{d^2x_2}{du_2^2} & \dfrac{d^2x_3}{du_2^2} \\ \dfrac{dx_1}{du_1} & \dfrac{dx_2}{du_1} & \dfrac{dx_3}{du_1} \\ \dfrac{dx_1}{du_2} & \dfrac{dx_2}{du_2} & \dfrac{dx_3}{du_2} \end{vmatrix}.$$

The quantities A_1, A_2, A_3 are the minors of these corresponding to the constituents of the first row in each. We find now readily

$$\frac{d\cos\theta_1}{du_1}=\frac{d}{du_1}\cdot\frac{A_1}{V}=-\frac{1}{V^3}\left[\left(E'_{11}E_{22}-E'_{12}E_{12}\right)\frac{dx_1}{du_1}+\left(E'_{12}E_{11}-E'_{11}E_{12}\right)\frac{dx_1}{du_2}\right]$$

$$\frac{d\cos\theta_1}{du_2}=\frac{d}{du_2}\cdot\frac{A_1}{V}=-\frac{1}{V^3}\left[\left(E'_{12}E_{22}-E'_{22}E_{12}\right)\frac{dx_1}{du_1}+\left(E'_{22}E_{11}-E'_{12}E_{12}\right)\frac{dx_1}{du_2}\right]$$

$$\frac{d\cos\theta_2}{du_1}=\frac{d}{du_1}\cdot\frac{A_2}{V}=-\frac{1}{V^3}\left[\left(E'_{11}E_{22}-E'_{12}E_{12}\right)\frac{dx_2}{du_1}+\left(E'_{12}E_{11}-E'_{11}E_{12}\right)\frac{dx_2}{du_2}\right]$$

$$\frac{d\cos\theta_2}{du_2}=\frac{d}{du_2}\cdot\frac{A_2}{V}=-\frac{1}{V^3}\left[\left(E'_{12}E_{22}-E'_{22}E_{12}\right)\frac{dx_2}{du_1}+\left(E'_{22}E_{11}-E'_{12}E_{12}\right)\frac{dx_2}{du_2}\right]$$

Substituting these values in the first factor in brackets in the expression for k, we have, after easy reductions, for this factor the value

$$\frac{E'_{11}E'_{22}E'^2_{12}}{V^4}\left(\frac{dx_1}{du_1}\frac{dx_2}{du_2}-\frac{dx_1}{du_2}\frac{dx_2}{du_1}\right)=A_3\frac{E'_{11}E'_{22}-E'^2_{12}}{V^4}=\cos\theta_3\frac{E'_{11}E'_{22}-E'^2_{12}}{V^3}.$$

The remaining factor in brackets becomes obviously

$$\frac{A_1^2+A_2^2+A_3^2}{V^3}=\frac{1}{V}.$$

So finally collecting all the terms we have

$$k=\frac{E'_{11}E'_{22}-E'^2_{12}}{V^4},$$

or in the ordinary notation,

$$k=\frac{E'\,G'-F'^2}{(EG-F^2)^2}$$

the well-known form for the measure of curvature. We have, then, the theorem that if $\cos\theta_1$, $\cos\theta_2$, $\cos\theta_3$ are the direction-cosines of the normal at any point x, y, z of a surface, the measure of curvature of the surface at this point is given by

$$k=\begin{vmatrix}\cos\theta_1, & \dfrac{d\cos\theta_1}{dx} & \dfrac{d\cos\theta_1}{dy} & \dfrac{d\cos\theta_1}{dz}\\ \cos\theta_2, & \dfrac{d\cos\theta_2}{dx} & \dfrac{d\cos\theta_2}{dy} & \dfrac{d\cos\theta_2}{dz}\\ \cos\theta_3, & \dfrac{d\cos\theta_3}{dx} & \dfrac{d\cos\theta_3}{dy} & \dfrac{d\cos\theta_3}{dz}\\ 0 & \cos\theta_1 & \cos\theta_2 & \cos\theta_3\end{vmatrix}$$

or again by

$$k = \frac{\sin\theta_1 \sin\theta_2}{\cos\theta_3} \begin{vmatrix} \frac{d\theta_1}{dx} & \frac{d\theta_1}{dy} & \frac{d\theta_1}{dz} \\ \frac{d\theta_2}{dx} & \frac{d\theta_2}{dy} & \frac{d\theta_2}{dz} \\ \cos\theta_1 & \cos\theta_2 & \cos\theta_3 \end{vmatrix}.$$

There are of course two other forms similar to this got by advancing the suffixes of θ. The bordered determinant whose value in the general case is K may in like manner be shown to be equal to

$$\frac{V'}{V^{n+2}},$$

where V' is the result of accenting all the letters in V just as has been done in the case of an ordinary surface and a unit sphere, so that $\frac{V'}{V^{n+2}}$ is the measure of curvature of a surface

$$\Phi(x_1 x_2 \ldots x_{n+1}) = 0$$

in M_{n+1}, the unit sphere being

$$Y_1^2 + Y_2^2 + \ldots + Y_{n+1}^2 = 1.$$

The correspondence between the surface $\Phi = 0$ and the n-dimensional unit sphere is of course just the same as in the ordinary case. The direction-cosines of the normal to Φ are

$$\frac{\Phi_1}{Q}, \quad \frac{\Phi_2}{Q}, \ldots \frac{\Phi_{n+1}}{Q}$$

the subscripts indicating differential coefficients and

$$Q = \sqrt{\sum_1^{n+1} \Phi_i^2}.$$

The required correspondence between the surface Φ and the sphere is produced by writing

$$Y_i = \frac{\Phi_i}{Q}.$$

Consider consecutive points on both surfaces: on the surface Φ these are given by

$$(x_1, x_2 \ldots x_{n+1}), \left(x_1 + \frac{dx_1}{du_i} du_i, \; x_2 + \frac{dx_2}{du_i} du_i \ldots\right)$$

and on the sphere the corresponding consecutive points are

$$(Y_k) \text{ and } \left(Y_k + \frac{dy_k}{du_i} du_i\right) \cdot$$

Consider two other points (ξ) and (η) lying in M_{n+1}, one near Φ and the other near the sphere. Denoting now by G and Γ the volumes of the small parallelopipedons formed by these two systems of points, we have as is well known (*vide* Beez, *Mathematische Annalen*, Vol. VI),

$$G = \begin{vmatrix} \xi_1 - x_1 & \xi_2 - x_2 & \cdots & \xi_{n+1} - x_{n+1} \\ \frac{dx_1}{du_1} & \frac{dx_2}{du_1} & \cdots & \frac{dx_{n+1}}{du_1} \\ \frac{dx_1}{du_2} & \frac{dx_2}{du_2} & \cdots & \frac{dx_{n+1}}{du_2} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{dx_1}{du_n} & \frac{dx_2}{du_n} & \cdots & \frac{dx_{n+1}}{du_n} \end{vmatrix} \prod_1^n du_i, \quad \Gamma = \begin{vmatrix} \eta_1 - y_1, & \eta_2 - y_2 & \cdots & \eta_{n+1} - y_{n+1} \\ \frac{dy_1}{du_1} & \frac{dy_2}{du_1} & \cdots & \frac{dy_{n+1}}{du_1} \\ \frac{dy_1}{du_2} & \frac{dy_2}{du_2} & \cdots & \frac{dy_{n+1}}{du_2} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{dy_1}{du_n} & \frac{dy_2}{du_n} & \cdots & \frac{dy_{n+1}}{du_n} \end{vmatrix} \prod_1^n du_i$$

The minors of these determinants corresponding to $(\xi - x)$ in the one and $(\eta - y)$ in the other are obviously the quantities A and B of the earlier part of the paper; and of course

$$\Sigma A^2 = V^2, \qquad \Sigma B^2 = U^2,$$

and

$$\frac{A_i}{V} = \frac{\Phi_i}{Q}, \qquad \frac{B_i}{U} = Y_i.$$

Call the distance between the points (x) and (ξ) δ, then expand the above determinant and divide through by $V.\delta.\prod_1^n du_i$; this gives

$$\frac{G}{\delta . V . \Pi du} = \Sigma \frac{A_i}{V} \cdot \frac{\xi_i - x_i}{\delta}.$$

Now $\left(\frac{A_i}{V}\right)$ are the direction-cosines of the normal to Φ at the point (x), and $\left(\frac{\xi_i - x_i}{\delta}\right)$ are the direction-cosines of the straight line joining (x) and (ξ). Denoting by θ the angle between the normal and this line $(x\xi)$ we have

$$\cos\theta = \sum_1^{n+1} \frac{A_i}{V} \cdot \frac{\xi_i - x_i}{\delta},$$

and consequently

$$V \cdot \prod_1^n du_i = \frac{G}{\delta \cos\theta} \cdot$$

G is the volume of the parallelopipedon, δ its slant height, and obviously $\delta \cos\theta$ its altitude; consequently $\frac{G}{\delta\cos\theta}$ is the base, or the element of the surface Φ is given by the quantity $V\cdot\prod_1^n du_i\cdot$

This is the value of dS that has been employed. The proof that I have given is similar to one given by Beez in the *Math. Ann.;* but as I obtained my proof before reading Beez's article, and as the two proofs differ considerably in the methods of working them out, I let mine stand just as I obtained it. It is obvious that for the correspondence between the sphere and the surface Φ required in order to find the value of the measure of curvature, it is only necessary to write

$$Y_i = \frac{A}{V}$$

and so

$$\frac{dy_i}{du_k} = \frac{1}{V^2}\left\{ V\frac{dA_i}{du_k} - A_i\frac{dV}{du_k}\right\}.$$

Consider the determinant B_1, *i. e.*

$$B_1 = \begin{vmatrix} \frac{dy_2}{du_1} & \frac{dy_2}{du_2} & \cdots & \frac{dy_2}{du_n} \\ \frac{dy_3}{du_1} & \frac{dy_3}{du_2} & \cdots & \frac{dy_3}{du_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{dy_{n+1}}{du_1} & \frac{dy_{n+1}}{du_2} & \cdots & \frac{dy_{n+1}}{du_n} \end{vmatrix}.$$

Substituting in this the value of each constituent as given by the above formula, we find after simple reductions,

$$B_1 = \frac{1}{V^{n+2}}\begin{vmatrix} V^2 & A_2 & A_3 & \cdots & A_{n+1} \\ \frac{1}{2}\frac{dV^2}{du_1} & \frac{dA_2}{du_1} & \frac{dA_3}{du_1} & \cdots & \frac{dA_{n+1}}{du_1} \\ \frac{1}{2}\frac{dV^2}{du_2} & \frac{dA_2}{du_2} & \frac{dA_3}{du_2} & \cdots & \frac{dA_{n+1}}{du_2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{2}\frac{dV^2}{du_n} & \frac{dA_2}{du_n} & \frac{dA_3}{du_n} & \cdots & \frac{dA_{n+1}}{du_n} \end{vmatrix}.$$

Now since
$$V^2 = \sum_1^{n+1} A_i^2,$$
$$V\frac{dV}{du_i} = A_1\frac{dA_1}{du_i} + A_2\frac{dA_2}{du_i} + \ldots + A_{n+1}\frac{dA_{n+1}}{du_i};$$

so multiplying the second column of the determinant by A_2, the third by A_3, etc., and subtracting the sum of the products from the first column, we have

$$\frac{A_1}{r_{n+2}}\begin{vmatrix} A_1 & A_2 & \ldots & A_{n+1} \\ \frac{dA_1}{du_1} & \frac{dA_2}{du_1} & \ldots & \frac{dA_{n+1}}{du_1} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{dA_1}{du_n} & \frac{dA_2}{du_n} & \ldots & \frac{dA_{n+1}}{du_n} \end{vmatrix}.$$

By definition of the quantities A_i we have identically

$$1 = \frac{1}{V^2}\begin{vmatrix} A_1 & A_2 & \ldots & A_{n+1} \\ \frac{dx_1}{du_1} & \frac{dx_2}{du_1} & \ldots & \frac{dx_{n+1}}{du_1} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{dx_1}{du_n} & \frac{dx_2}{du_n} & \ldots & \frac{dx_{n+1}}{du_n} \end{vmatrix};$$

also
$$\sum_{i=1}^{i=n+1} A_i\frac{dx_i}{du_k} = 0, \quad k = 1, 2, 3 \ldots n;$$

differentiate this and write as above
$$E'_{ik} = \left(A_1\frac{d^2x_1}{du_idu_k} + A_2\frac{d^2x_2}{du_idu_k} + \ldots\right)$$
$$= -\left(\frac{dA_1}{du_i}\frac{dx_1}{du_k} + \frac{dA_2}{du_i}\frac{dx_2}{du_k} + \ldots\right).$$

Now multiplying together the last two determinants, we find readily for the measure of curvature the value
$$k = \frac{U}{V^{n+2}},$$

where U is the determinant formed out of the accented letters E'_{ik} in the same manner that V is formed from E_{ik}.

It is not necessary to go into the proof that the determinant

$$\begin{vmatrix} \frac{A_1}{V} & \frac{d}{du_1}\frac{A_1}{V} & \cdots & \frac{d}{du_n}\frac{A_1}{V} \\ \frac{A_2}{V} & \frac{d}{du_1}\frac{A_2}{V} & \cdots & \frac{d}{du_n}\frac{A_2}{V} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{A_{n+1}}{V} & \frac{d}{du_1}\frac{A_{n+1}}{V} & \cdots & \frac{d}{du_n}\frac{A_{n+1}}{V} \\ 0 & \frac{A_1}{V} & \cdots & \frac{A_{n+1}}{V} \end{vmatrix}$$

reduces in this case, as in the simple one of $n = 2$, to the ratio

$$\frac{U}{V^{n+2}}$$

and consequently that this bordered determinant gives the measure of curvature for the n-dimensional surface $\Phi = 0$ in M_{n+1}.

In seeking the expression for the radii of curvature of the surface $\Phi = 0$ at any point, it will be sufficient to consider the case of $n + 1 = 4$.

Denote by $\xi_1, \xi_2, \xi_3, \xi_4$ the coördinates of any point on the normal to $\Phi = 0$ at the point x_1, x_2, x_3, x_4; then

$$\xi_i = x_i + A_i\lambda, \qquad i = 1, 2, 3, 4.$$

Suppose (ξ) to be the point where this normal meets the consecutive normal, then

$$dx_i + A_i d\lambda + \lambda dA_i = 0;$$

but

$$dx_i = a_1^{(i)}du_1 + a_2^{(i)}du_2 + a_3^{(i)}du_3,$$

and similarly we may write

$$dA_i = A_1^{(i)}du_1 + A_2^{(i)}du_2 + A_3^{(i)}du_3;$$

consequently

$$(a_1^{(i)} + \lambda A_1^{(i)})\, du_1 + \ldots + A_i d\lambda = 0, \qquad i = 1, 2, 3, 4.$$

Eliminating du_1, du_2, du_3 and $d\lambda$ from these we obtain

$$\begin{vmatrix} a_1^{(1)} + \lambda A_1^{(1)}, & a_2^{(1)} + \lambda A_2^{(1)}, & a_3^{(1)} + \lambda A_3^{(1)}, & A_1 \\ a_1^{(2)} + \lambda A_1^{(2)}, & a_2^{(2)} + \lambda A_2^{(2)}, & a_3^{(2)} + \lambda A_3^{(2)}, & A_2 \\ a_1^{(3)} + \lambda A_1^{(3)}, & a_2^{(3)} + \lambda A_2^{(3)}, & a_3^{(3)} + \lambda A_3^{(3)}, & A_3 \\ a_1^{(4)} + \lambda A_1^{(4)}, & a_2^{(4)} + \lambda A_2^{(4)}, & a_3^{(4)} + \lambda A_3^{(4)}, & A_4 \end{vmatrix} = 0.$$

Calling ρ the radius of curvature, we have $\rho^2 = \Sigma(\xi_i - x_i)^2 = \lambda^2 V^2$, and consequently

$$\lambda = \frac{\rho}{V}\cdot$$

Substituting this in the previous equations and making one simple reduction, the equation becomes

$$\begin{vmatrix} E_{11}V + \rho\Sigma A_1^{(i)}a_1^{(i)}, & E_{12}V + \rho\Sigma A_1^{(i)}a_2^{(i)}, & E_{13}V + \rho\Sigma A_1^{(i)}a_3^{(i)} \\ E_{21}V + \rho\Sigma A_2^{(i)}a_1^{(i)}, & E_{22}V + \rho\Sigma A_2^{(i)}a_2^{(i)}, & E_{23}V + \rho\Sigma A_2^{(i)}a_3^{(i)} \\ E_{31}V + \rho\Sigma A_3^{(i)}a_1^{(i)}, & E_{32}V + \rho\Sigma A_3^{(i)}a_2^{(i)}, & E_{33}V + \rho\Sigma A_3^{(i)}a_3^{(i)} \end{vmatrix} = 0.$$

Introduce here the accented letters E'_{jk}, viz.

$$E'_{jk} = A_1 b_{jk}^{(1)} + A_2 b_{jk}^{(2)} + A_3 b_{jk}^{(3)} + A_4 b_{jk}^{(4)}$$

in which

$$b_{kk}^{(i)} = \frac{d^2x_i}{du_k^2} = \frac{da_k^{(i)}}{du_k}$$

$$b_{jk}^{(i)} = \frac{d^2x_i}{du_j du_k} = \frac{da_j^{(i)}}{du_k} = \frac{da_k^{(i)}}{du_j}\cdot$$

Differentiate the identity

$$\sum_{j=1}^{j=4} A_j a_1^{(j)} = 0$$

and we obtain

$$\sum_{j=1}^{j=4} a_1^{(j)} A_i^{(j)} = -\sum_{j=1}^{j=4} A_j b_{1i}^{(j)}$$

There are in all six equations of this kind; the quantities on the left hand side of each of these equations are the coefficients of ρ in the above cubic equation, while the quantities on the right are equal to

$$-E_{11},\ -E_{22},\ -E_{33},\ -E_{12},\ -E_{31},\ -E_{23}.$$

The cubic equation for the determination of ρ becomes now

$$\begin{vmatrix} E'_{11}\rho - E_{11}V, & E'_{12}\rho - E_{12}V, & E'_{13}\rho - E_{13}V \\ E'_{21}\rho - E_{21}V, & E'_{22}\rho - E_{22}V, & E'_{23}\rho - E_{23}V \\ E'_{31}\rho - E_{31}V, & E'_{32}\rho - E_{32}V, & E'_{33}\rho - E_{33}V \end{vmatrix} = 0.$$

The equation for the determination of the radii of curvature of $\Phi = 0$ in M_{n+1} is obviously

$$\begin{vmatrix} E'_{11}\rho - E_{11}V, & E'_{12}\rho - E_{12}V & \ldots & E'_{1n}\rho - E_{1n}V \\ E'_{21}\rho - E_{21}V, & E'_{22}\rho - E_{22}V & \ldots & E'_{2n}\rho - E_{2n}V \\ \cdots & \cdots & \cdots & \cdots \\ E'_{n1}\rho - E_{n1}V, & E'_{n2}\rho - E_{n2}V & \ldots & E'_{nn}\rho - E_{nn}V \end{vmatrix} = 0.$$

The coefficient of ρ^n in this is U, where as before

$$U^2 = |E'_{jk}|, \qquad V^2 = |E_{jk}|,$$

and the constant term is V^{n+2}. Consequently denoting by $\rho_1, \rho_2 \ldots \rho_n$ the roots of this equation, we have

$$\frac{1}{\rho_1 \rho_2 \cdots \rho_n} = \frac{U}{V^{n+2}},$$

the value of the measure of curvature.

The equation

$$U = V^{n+2}$$

is the differential equation of all surfaces developable upon the n-dimensional sphere. The result of equating the coefficient of ρ to zero, *i. e.*

$$\rho_1 + \rho_2 + \ldots + \rho_n = 0$$

corresponds to a class of surfaces similar to surfaces of minimum area.

Denoting by $\alpha_1, \alpha_2, \alpha_3$ the direction-cosines of a normal to the surface $\Phi(x_1 x_2 x_3) = 0$ and by $\beta_1, \beta_2, \beta_3$ the direction-cosines to the second surface at the corresponding point, write

$$\begin{array}{llll}
l_1, & l_2, & l_3 = & \dfrac{d\alpha_1}{du_1}, \quad \dfrac{d\alpha_2}{du_1}, \quad \dfrac{d\alpha_3}{du_1} \\[2ex]
m_1, & m_2, & m_3 = & \dfrac{d\alpha_1}{du_2}, \quad \dfrac{d\alpha_2}{du_2}, \quad \dfrac{d\alpha_3}{du_2} \\[2ex]
\lambda_1, & \lambda_2, & \lambda_3 = & \dfrac{d\beta_1}{du_1}, \quad \dfrac{d\beta_2}{du_1}, \quad \dfrac{d\beta_3}{du_1} \\[2ex]
\mu_1, & \mu_2, & \mu_3 = & \dfrac{d\beta_1}{du_2}, \quad \dfrac{d\beta_2}{du_2}, \quad \dfrac{d\beta_3}{du_2}.
\end{array}$$

Then obviously

$$\begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{vmatrix} = Vk,$$

and, calling K the measure of curvature of the second surface,

$$\begin{vmatrix} \beta_1 & \beta_2 & \beta_3 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \end{vmatrix} = UK.$$

Take now the determinant

$$\begin{vmatrix} \beta_1 & \beta_2 & \beta_3 & 0 & 0 \\ \lambda_1 & \lambda_2 & \lambda_3 & 0 & 0 \\ \mu_1 & \mu_2 & \mu_3 & 0 & 0 \\ \frac{dy_1}{du_1} & \frac{dy_2}{du_1} & \frac{dy_3}{du_1} & \frac{E'}{V}, & \frac{F'}{V} \\ \frac{dy_1}{du_2} & \frac{dy_2}{du_2} & \frac{dy_3}{du_2} & \frac{F'}{V}, & \frac{G'}{V} \end{vmatrix}.$$

This is obviously

$$= V^2 k \,.\, UK.$$

Substituting for $\frac{dy}{du}$, &c., their values

$$\frac{dy}{du} = \frac{dy}{dx_1}\frac{dx_1}{du} + \frac{dy}{dx_2}\frac{dx_2}{du} + \frac{dy}{dx_3}\frac{dx_3}{du}, \quad \&c.$$

this determinant is readily seen to be equal to the product

$$V \begin{vmatrix} \lambda_1 & \mu_1 & \beta_1 & \frac{dy_1}{dx_1} & \frac{dy_1}{dx_2} & \frac{dy_1}{dx_3} \\ \lambda_2 & \mu_2 & \beta_2 & \frac{dy_2}{dx_1} & \frac{dy_2}{dx_2} & \frac{dy_2}{dx_3} \\ \lambda_3 & \mu_3 & \beta_3 & \frac{dy_3}{dx_1} & \frac{dy_3}{dx_2} & \frac{dy_3}{dx_3} \\ 0 & 0 & 0 & \alpha_1 & \alpha_2 & \alpha_3 \\ 0 & 0 & 0 & l_1 & l_2 & l_3 \\ 0 & 0 & 0 & m_1 & m_2 & m_3 \end{vmatrix}.$$

This is obviously $= V^2 UkK$, so that no results of any consequence can be obtained by bordering the functional determinant

$$\frac{d\,(y_1 y_2 y_3)}{d\,(x_1 x_2 x_3)}$$

in this manner. Take the determinant

$$\begin{vmatrix} \frac{dx_1}{du_2} & \frac{dx_1}{du_1} & \frac{dy_1}{dx_1} & \frac{dy_1}{dx_2} & \frac{dy_1}{dx_3} \\ \frac{dx_2}{du_2} & \frac{dx_2}{du_1} & \frac{dy_2}{dx_1} & \frac{dy_2}{dx_2} & \frac{dy_2}{dx_3} \\ \frac{dx_3}{du_2} & \frac{dx_3}{du_1} & \frac{dy_3}{dx_1} & \frac{dy_3}{dx_2} & \frac{dy_3}{dx_3} \\ 0 & 0 & \frac{dx_1}{du_1} & \frac{dx_2}{du_1} & \frac{dx_3}{du_1} \\ 0 & 0 & \frac{dx_1}{du_2} & \frac{dx_2}{du_2} & \frac{dx_3}{du_2} \end{vmatrix}, = \Delta$$

and multiply this by the determinant giving V^2, bordered in a suitable manner. The product is readily found to give

$$
-V^2 \begin{vmatrix} \frac{dx_1}{du_1} & \frac{dx_2}{du_1} & \frac{dx_3}{du_1} \\ \frac{dx_1}{du_2} & \frac{dx_2}{du_2} & \frac{d\,x_3}{du_2} \\ \Sigma A \frac{dy_1}{dx} & \Sigma A \frac{dy_2}{dx} & \Sigma A \frac{dy_3}{dx} \end{vmatrix}
$$

$$
= -V^2 \left\{ A_1 \Sigma A_i \frac{dy_1}{dx_i} + A_2 \Sigma A_i \frac{dy_2}{dx_i} + A_3 \Sigma A_i \frac{dy_3}{dx_i} \right\}
$$

$$
= - V^2 \Sigma A_k \Sigma A_i \frac{dy_k}{dx_i}.
$$

Substituting for $\frac{dy}{dx}$ the value

$$
\frac{dy}{du_1}\frac{du_1}{dx} + \frac{dy}{du_2}\frac{du_2}{dx},
$$

and dropping the factor V^2, we have

$$
-\Delta = \Sigma A_i \frac{du_1}{dx_i} \Sigma A_i \frac{dy_i}{du_1} + \Sigma A_i \frac{du_2}{dx_i} \Sigma A_i \frac{dy_i}{du_2}.
$$

The first of the factors in this expression is proportional to the cosine of the angle between the normals to the surfaces $\Phi = 0$ and $u_1 = \text{const.}$ The second factor is proportional to the cosine of the angle between the normal to $\Phi = 0$ and the tangent to the curve $u_1 = \text{const.}$ traced on the second surface—say $\Psi = 0$. The first factor of the last term is proportional to the cosine of the angle between the normals to $\Phi = 0$ and $u_2 = \text{const.}$, and the last factor is proportional to the cosine of the angle between the normal to $\Phi = 0$ and the tangent to the curve $u_2 = \text{const.}$, $\Psi = 0$. Call these angles $\theta_1, \phi_1, \theta_2, \phi_2$, and write

$$
\Sigma \left(\frac{dy_i}{du_1}\right)^2 = \mathfrak{E}, \quad \Sigma \left(\frac{dy_i}{du_2}\right)^2 = \mathfrak{G}, \quad \Sigma \frac{dy_i}{du_1}\frac{dy_i}{du_2} = \mathfrak{F},
$$

$$
\Sigma \left(\frac{du_1}{dx_i}\right)^2 = L, \quad \Sigma \left(\frac{du_2}{dx_i}\right)^2 = M; \text{ we have then at once}
$$

$$
-\Delta = V^2 \{ \mathfrak{E} L \cos\theta_1 \cos\phi_1 + \mathfrak{G} M \cos\theta_2 \cos\phi_2 \}.
$$

If the surfaces $u = \text{const.}$ intersect $\Phi = 0$ orthogonally, we have $\cos\theta_1 = \cos\theta_2 = 0$ and so $\Delta = 0$. If $\cos\phi_1 = \cos\phi_2 = 0$, all the normals to the surface $\Phi = 0$ will be parallel to the normals at the corresponding points of $\Psi = 0$.

The determinant

$$\frac{d\,(y_1 y_2 \ldots y_{n+1})}{d\,(x_1 x_2 \ldots x_{n+1})}$$

bordered as above will lead to a similar result, viz.

$$-\Delta = V^2 \Sigma \mathfrak{H}_i L_i \cos\theta_i \cos\phi_i .$$

Denote by xyz a point on the first surface, then

$$\frac{1}{\nu}\begin{vmatrix} x & y & z \\ \frac{dx}{du} & \frac{dy}{du} & \frac{dz}{du} \\ \frac{dx}{dv} & \frac{dy}{dv} & \frac{dz}{dv} \end{vmatrix} = \cos\theta ,$$

θ denoting the angle between the radius vector from the origin to (xyz) and the normal, and $\nu = \sqrt{x^2 + y^2 + z^2}$. Denote by ρ and ψ the corresponding radius vector and angle on the second surface, then

$$\frac{1}{\rho}\begin{vmatrix} \xi & \eta & \zeta \\ \frac{d\xi}{du} & \frac{d\eta}{du} & \frac{d\zeta}{du} \\ \frac{d\xi}{dv} & \frac{d\eta}{dv} & \frac{d\zeta}{dv} \end{vmatrix} = \cos\psi .$$

Multiply together the two determinants

$$\begin{vmatrix} \xi & \frac{d\xi}{dx} & \frac{d\xi}{dy} & \frac{d\xi}{dz} \\ \eta & \frac{d\eta}{dx} & \frac{d\eta}{dy} & \frac{d\eta}{dz} \\ \zeta & \frac{d\zeta}{dx} & \frac{d\zeta}{dy} & \frac{d\zeta}{dz} \\ 0 & a & b & c \end{vmatrix} \cdot \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{dx}{du} & \frac{dy}{du} & \frac{dz}{du} \\ 0 & \frac{dx}{dv} & \frac{dy}{dv} & \frac{dz}{dv} \\ 0 & x & y & z \end{vmatrix}$$

a, b, c being the direction-cosines of the normal to the first surface at (xyz). The product of these two is readily found to be

$$= \nu \cos\theta \begin{vmatrix} \xi & \eta & \zeta \\ \frac{d\xi}{du} & \frac{d\eta}{du} & \frac{d\zeta}{du} \\ \frac{d\xi}{dv} & \frac{d\eta}{dv} & \frac{d\zeta}{dv} \end{vmatrix} = \nu \cos\theta \,.\, \rho \cos\psi$$

therefore

$$\cos\psi = \frac{1}{\rho}\begin{vmatrix} \xi & \frac{d\xi}{dx} & \frac{d\xi}{dy} & \frac{d\xi}{dz} \\ \eta & \frac{d\eta}{dx} & \frac{d\eta}{dy} & \frac{d\eta}{dz} \\ \zeta & \frac{d\zeta}{dx} & \frac{d\zeta}{dy} & \frac{d\zeta}{dz} \\ 0 & a & b & c \end{vmatrix}.$$

The ratio between the corresponding elements of area on two corresponding surfaces (xyz), $(\xi\eta\zeta)$ has been shown to be $=$.

$$\begin{vmatrix} \alpha & \frac{d\xi}{dx} & \frac{d\xi}{dy} & \frac{d\xi}{dz} \\ \beta & \frac{d\eta}{dx} & \frac{d\eta}{dy} & \frac{d\eta}{dz} \\ \gamma & \frac{d\zeta}{dx} & \frac{d\zeta}{dy} & \frac{d\zeta}{dz} \\ 0 & a & b & c \end{vmatrix} = -\frac{U}{V}.$$

This can be given in a different form in the case when the surfaces

$$\xi = F_1(xyz)$$
$$\eta = F_2(xyz)$$
$$\zeta = F_3(xyz)$$

are orthogonal. The coördinates of a point on the second surface may be considered as given by the intersection of the three surfaces

$$F_1 = \text{const.},\quad F_2 = \text{const.},\quad F_3 = \text{const.}$$

The conditions for orthogonality of these surfaces are, writing ξ, η, ζ instead of F_1, F_2, F_3,

$$\frac{d\xi}{dx}\frac{d\eta}{dx} + \frac{d\xi}{dy}\frac{d\eta}{dy} + \frac{d\xi}{dz}\frac{d\eta}{dz} = 0$$
$$\frac{d\eta}{dx}\frac{d\zeta}{dx} + \frac{d\eta}{dy}\frac{d\zeta}{dy} + \frac{d\eta}{dz}\frac{d\zeta}{dz} = 0$$
$$\frac{d\zeta}{dx}\frac{d\xi}{dx} + \frac{d\zeta}{dy}\frac{d\xi}{dy} + \frac{d\zeta}{dz}\frac{d\xi}{dz} = 0.$$

Find the product of the last given determinant and the functional determinant

$$\frac{d(\xi\eta\zeta)}{d(xyz)}$$

bordered by 1, 0, 0, 0 for its first row and by the same for its first column.

Write

$$\left(\frac{d\xi}{dx}\right)^2+\left(\frac{d\xi}{dy}\right)^2+\left(\frac{d\xi}{dz}\right)^2=\Xi^2$$
$$\left(\frac{d\eta}{dx}\right)^2+\left(\frac{d\eta}{dy}\right)^2+\left(\frac{d\eta}{dz}\right)^2=H^2$$
$$\left(\frac{d\zeta}{dx}\right)^2+\left(\frac{d\zeta}{dy}\right)^2+\left(\frac{d\zeta}{dz}\right)^2=Z^2.$$

We have, then, for the required product

$$\begin{vmatrix} \alpha & \beta & \gamma & 0 \\ \Xi^2 & 0 & 0 & \Xi\cos\theta_1 \\ 0 & H^2 & 0 & H\cos\theta_2 \\ 0 & 0 & Z^2 & Z\cos\theta_3 \end{vmatrix}$$

where θ_1, θ_2, θ_3 are the angles which the normal to the original surface makes with the normals to the surfaces ξ, η, ζ at their common point of intersection. Expanding this we have

$$\Xi HZ\left\{HZ\alpha\cos\theta_1+Z\Xi\beta\cos\theta_2+\Xi H\gamma\cos\theta_3\right\}$$

and consequently

$$\frac{U}{V}=\frac{-\Xi HZ}{\dfrac{d(\xi\eta\zeta)}{d(xyz)}}\left\{HZ\alpha\cos\theta_1+Z\Xi\beta\cos\theta_2+\Xi H\gamma\cos\theta_3\right\}$$

or

$$=\frac{-\Xi^2H^2Z^2}{\dfrac{d(\xi\eta\zeta)}{d(xyz)}}\left\{\frac{\alpha\cos\theta_1}{\Xi}+\frac{\beta\cos\theta_2}{H}+\frac{\gamma\cos\theta_3}{Z}\right\}$$

If we denote for the moment by

$$\begin{matrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{matrix}$$

the direction-cosines of the normals to the surfaces ξ, η, ζ respectively, we have

$$\frac{d(\xi\eta\zeta)}{d(xyz)}=\Xi HZ\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}=\Xi HZ;$$

and consequently

$$\frac{d\Sigma}{dS}, = \frac{U}{V}, = -\Xi HZ \left\{ \frac{\alpha \cos \theta_1}{\Xi} + \frac{\beta \cos \theta_2}{H} + \frac{\gamma \cos \theta_3}{Z} \right\}.$$

Of course a similar formula applies for a surface of n dimensions in M_{n+1}.

I will now work out a few simple properties of curves in a space of more than three dimensions. The results obtained are (with one exception which is referred to below) I believe new. For simplicity, at first consider only a space of four dimensions—M_4. A curve in M_4 in the ordinary acceptation of that term will depend only on one parameter—it may either be given by the values of its coördinates expressed as functions of this parameter, or as the intersection of three 3-dimensional surfaces, say

$$\Phi_1 = 0, \quad \Phi_2 = 0, \quad \Phi_3 = 0.$$

Let u denote the parameter. First obtain the equation of the osculating flat space of three dimensions at any point of the curve. Four points on the curve will determine the osculating 3-flat. Take then four points on the curve, say A, B, C, D. The coördinates are

A	B	C	D
x_1	$x_1 + \Delta x_1$	$x_1 + 2\Delta x_1 + \Delta^2 x_1$	$x_1 + 3\Delta x_1 + 3\Delta^2 x_1 + \Delta^3 x_1$
x_2	$x_2 + \Delta x_2$	$x_2 + 2\Delta x_2 + \Delta^2 x_2$	$x_2 + 3\Delta x_2 + 3\Delta^2 x_2 + \Delta^3 x_2$
x_3	$x_3 + \Delta x_3$	$x_3 + 2\Delta x_3 + \Delta^2 x_3$	$x_3 + 3\Delta x_3 + 3\Delta^2 x_3 + \Delta^3 x_3$
x_4	$x_4 + \Delta x_4$	$x_4 + 2\Delta x_4 + \Delta^2 x_4$	$x_4 + 3\Delta x_4 + 3\Delta^2 x_4 + \Delta^3 x_4$

Now if $(\xi_1, \xi_2, \xi_3, \xi_4)$ is any point in the 3-flat passing through (x), its equation is

$$a_1(\xi_1 - x_1) + a_2(\xi_2 - x_2) + a_3(\xi_3 - x_3) + a_4(\xi_4 - x_4) = 0.$$

Substitute for (ξ) the coördinates of B and we have

$$a_1\Delta x_1 + a_2\Delta x_2 + a_3\Delta x_3 + a_4\Delta x_4 = 0.$$

In like manner substituting the coördinates of C and D in the place of (ξ), and using this last result to reduce the result of substituting the coördinates of C, and similarly reducing in the case of the point D, we have

$$\Sigma a\Delta^2 x = 0 \qquad \Sigma a\Delta^3 x = 0.$$

Eliminating (a) from the four equations,

$$\Sigma a(\xi - x) = 0 \qquad \Sigma a\Delta x = 0$$
$$\Sigma a\Delta^2 x = 0 \qquad \Sigma a\Delta^3 x = 0,$$

passing to the limit and dividing through by du^3, we have as the equation of the osculating 3-flat at the point (x) of the curve Φ

$$\begin{vmatrix} \xi_1 - x_1 & \xi_2 - x_2 & \xi_3 - x_3 & \xi_4 - x_4 \\ \frac{dx_1}{du} & \frac{dx_2}{du} & \frac{dx_3}{du} & \frac{dx_4}{du} \\ \frac{d^2x_1}{du^2} & \frac{d^2x_2}{du^2} & \frac{d^2x_3}{du^2} & \frac{d^2x_4}{du^2} \\ \frac{d^3x_1}{du^3} & \frac{d^3x_2}{du^3} & \frac{d^3x_3}{du^3} & \frac{d^3x_4}{du^3} \end{vmatrix} = 0 .$$

Of course in exactly the same manner we obtain

$$\begin{vmatrix} \xi_1 - x_1, & \xi_2 - x_2 & \dots & \xi_n - x_n \\ \frac{dx_1}{du} & \frac{dx_2}{du} & \dots & \frac{dx_n}{du} \\ \dots & \dots & \dots & \dots \\ \frac{d^{n-1}x_1}{du^{n-1}} & \frac{d^{n-1}x_2}{du^{n-1}} & \dots & \frac{d^{n-1}x_n}{du^{n-1}} \end{vmatrix} = 0$$

as the osculating $(n-1)$-flat to a curve in M_n. In a similar manner we obtain for the equations of the osculating 2-flat (*i. e.* osculating plane) to the point (x) on the curve Φ in M_4

$$\begin{Vmatrix} \xi_1 - x_1 & \xi_2 - x_2 & \xi_3 - x_3 & \xi_4 - x_4 \\ \frac{dx_1}{du} & \frac{dx_2}{du} & \frac{dx_3}{du} & \frac{dx_4}{du} \\ \frac{d^2x_1}{du^2} & \frac{d^2x_2}{du^2} & \frac{d^2x_3}{du^2} & \frac{d^2x_4}{du^2} \end{Vmatrix} = 0 .$$

And generally for the osculating k-flat to a curve in M_n we find without difficulty

$$\begin{Vmatrix} \xi_1 - x_1 & \xi_2 - x_2 & \dots & \xi_n - x_n \\ \frac{dx_1}{du} & \frac{dx_2}{du} & \dots & \frac{dx_n}{du} \\ \dots & \dots & \dots & \dots \\ \frac{d^kx_1}{du^k} & \frac{d^kx_2}{du^k} & \dots & \frac{d^kx_n}{du^k} \end{Vmatrix} = 0 .$$

The equations of the tangent to the curve Φ at the point (x) are of course

$$\begin{Vmatrix} \xi_1 - x_1 & \xi_2 - x_2 & \xi_3 - x_3 & \xi_4 - x_4 \\ \frac{dx_1}{du} & \frac{dx_2}{du} & \frac{dx_3}{du} & \frac{dx_4}{du} \end{Vmatrix} = 0 .$$

The direction-cosines of the perpendicular to the osculating 3-flat are

$$\alpha_1,\ \beta_1,\ \gamma_1,\ \eta_1 = \frac{1}{Q}\begin{vmatrix} \frac{dx_2}{du} & \frac{dx_3}{du} & \frac{dx_4}{du} \\ \frac{d^2x_2}{du^2} & \frac{d^2x_3}{du^2} & \frac{d^2x_4}{du^2} \\ \frac{d^3x_2}{du^3} & \frac{d^3x_3}{du^3} & \frac{d^3x_4}{du^3} \end{vmatrix},\ \text{\&c.}$$

where of course

$$Q = \begin{vmatrix} \Sigma\left(\frac{dx}{du}\right)^2, & \Sigma\frac{dx}{du}\frac{d^2x}{du^2}, & \Sigma\frac{dx}{du}\frac{d^3x}{du^3} \\ \Sigma\frac{dx}{du}\frac{d^2x}{du^2}, & \Sigma\left(\frac{d^2x}{du^2}\right)^2, & \Sigma\frac{d^2x}{du^2}\frac{d^3x}{du^3} \\ \Sigma\frac{dx}{du}\frac{d^3x}{du^3}, & \Sigma\frac{d^2x}{du^2}\frac{d^3x}{du^3}, & \Sigma\left(\frac{d^3x}{du^3}\right)^2 \end{vmatrix}$$

Suppose we make s the independent variable, then

$$\Sigma\left(\frac{dx}{ds}\right)^2 = 1$$

and

$$\Sigma\frac{dx}{ds}\frac{d^2x}{ds^2} = \frac{1}{2}\,d\Sigma\left(\frac{dx}{ds}\right)^2 = 0\,,$$

$$\Sigma\frac{dx}{ds}\frac{d^3x}{ds^3} = d\Sigma\frac{dx}{ds}\frac{d^2x}{ds^2} - d\Sigma\left(\frac{dx}{ds}\right)^2 = -\,d\Sigma\left(\frac{dx}{ds}\right)^2,$$

$$\Sigma\frac{d^2x}{ds^2}\frac{d^3x}{ds^3} = \frac{1}{2}\,d\Sigma\left(\frac{d^2x}{ds^2}\right)^2.$$

Therefore

$$Q = \begin{vmatrix} 1 & 0 & -\Sigma\left(\frac{d^2x}{ds^2}\right)^2 \\ 0 & \Sigma\left(\frac{d^2x}{ds^2}\right)^2, & \frac{1}{2}\,d\Sigma\left(\frac{d^2x}{ds^2}\right)^2 \\ -\Sigma\left(\frac{d^2x}{ds^2}\right)^2, & \frac{1}{2}\,d\Sigma\left(\frac{d^2x}{ds^2}\right)^2, & \Sigma\left(\frac{d^3x}{ds^3}\right)^2 \end{vmatrix} = \Sigma\left(\frac{d^2x}{ds^2}\right)^2\Sigma\left(\frac{d^3x}{ds^3}\right)^2 - \Sigma\left(\frac{d^2x}{ds^2}\right)^3.$$

It would not be difficult to find the radius of curvature of the curve at any point by introducing the osculating 3-dimensional sphere and finding its intersection with the osculating 3-flat giving an osculating 2-dimensional sphere. It

is not worth while doing this however, as the reductions are rather long, and as the same thing has been done, in a different manner, by G. E. A. Brunel in Vol. XIX (page 42) of the *Mathematische Annalen.* M. Brunel's formula is, changing his notation slightly,

$$\frac{1}{R_p^2} = \frac{1}{N_1} \cdot \frac{N_{p+1} N_{p-1}}{N_p^2}$$

where

$$N_p = \begin{Vmatrix} x_1' & x_2' & \cdots & x_n' \\ x_1'' & x_2'' & \cdots & x_n'' \\ \vdots & & & \\ x_1^{(p)} & x_2^{(p)} & \cdots & x_n^{(p)} \end{Vmatrix}^2$$

the accents denoting differential coefficients with respect to s.